Applied Mathematics and Computation 210 (2009) 169–176

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A variational approach to $(m+1)$-dimensional n-order field

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ARTICLE INFO

Keywords:
Multisymplectic scheme
Variational method

ABSTRACT

We present a method to construct the multisymplectic formulation for $(m+1)$-dimensional n-order field based on the variational principle. Then we derive the multisymplectic conservation law and the local energy and momentum conservation laws for $(m+1)$-dimensional n-order field. The centred box discretization of the multisymplectic formulation for $(m+1)$-dimensional n-order field is proved to be a multisymplectic scheme. As an example, the process of constructing the multisymplectic formulation for the KP equation is given.

1. Introduction

Many PDEs can be reformulated in Lagrangian setting. For $(1+1)$-dimensional first-order field, i.e., the Lagrangian density $\mathscr{L}$ depends on the state variables and their first-order partial derivatives and the state variables involve the time and 1 space variables, Bridges introduced the multisymplectic formulation to prove the existence of the multisymplectic conservation law and the local energy and momentum conservation laws. At the same time, he also proved that the centred box discretization keeps the discrete multisymplectic conservation law [1]. In this paper, our purpose is to generalize the Bridges multisymplectic formulation to $(m+1)$-dimensional n-order field.

In Lagrangian setting, taking the variation of the action function, we can directly derive the Euler–Lagrange equation for $(m+1)$-dimensional n-order field. When the boundary conditions are not considered, based on the "boundary part" of the functional derivative of the action we introduce a class of generalized canonical momenta and obtain the canonical multisymplectic structures for $(m+1)$-dimensional n-order field in Hamiltonian setting. At the same time, introducing a generalized Hamiltonian function we reformulate the Euler–Lagrange equation as a canonical Hamiltonian formulation. Writing the canonical Hamiltonian formulation as a matrix equation, we obtain the multisymplectic formulation for $(m+1)$-dimensional n-order field. The multisymplectic formulation is interesting for several reasons [2–4] and we can take it as a starting point for development of numerical methods for multisymplectic Hamiltonian PDEs [5]. Based on the multisymplectic formulation for $(m+1)$-dimensional n-order field, we directly derive that there exists a natural multisymplectic conservation law and local energy and momentum conservation laws for Hamiltonian PDEs. For first-order field, the centred box discretization of the multisymplectic formulation has been proved to be a multisymplectic scheme [1,6]. In this paper, we prove that the centred box discretization of the multisymplectic formulation for $(m+1)$-dimensional n-order field also keeps the discrete multisymplectic conservation law.

2. The multisymplectic formulation for $(m+1)$-dimensional n-order field

For $(m+1)$-dimensional n-order field, the corresponding Lagrangian density is

$$\mathscr{L} = \mathscr{L}(u_k) \quad (k \in \Omega), \tag{1}$$

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2008.12.065

where Ω is

$$\left\{\begin{array}{l} 0, x_0, x_1, x_2, \ldots, x_m, \\ x_0x_0, x_0x_1, x_0x_2, \ldots, x_0x_m, x_1x_0, x_1x_1, x_1x_2, \ldots, x_1x_m, \ldots, x_mx_0, x_mx_1, x_mx_2, \ldots, x_mx_m, \\ x_0x_0x_0, x_0x_0x_1, x_0x_0x_2, \ldots, x_0x_0x_m, \ldots, x_mx_mx_0, x_mx_mx_1, x_mx_mx_2, \ldots, x_mx_mx_m, \\ \ldots \end{array}\right\}, \tag{2}$$

x_0 is the time variable, and $x_1, x_2, \ldots, x_m$ are the space variables. We suppose that $u_{x_ix_j}$ and $u_{x_jx_i}$ are independent from each other (or different order of partial derivation for u makes independent variables).

The action function for $(m+1)$-dimensional n-order field is defined as

$$S = \int \cdots \int \mathscr{L}(u_k)\, dx_0\, dx_1\, dx_2 \cdots dx_m \quad (k \in \Omega). \tag{3}$$

Using integration by parts, we can obtain the variation of the action function as follows:

$$\begin{aligned} \delta S = {} & \int \cdots \int \left[\sum_{i\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_i}\right)_{i'}\right] du\, dx_0\, dx_1\, dx_2 \cdots dx_m \\ & + \int \cdots \int \left\{\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}\left[\sum_{i,j\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_{jx_\theta i}}\right)_{i'} du_j\right]\right\} dx_0\, dx_1\, dx_2 \cdots dx_m, \end{aligned} \tag{4}$$

where ℓ is the times of partial derivative in i. i' is the inverse order of i (e.g., if i is $x_1x_2x_3$, then i' is $x_3x_2x_1$).

By the variational principle which states that $\delta S = 0$, we derive the Euler–Lagrange equation (5) for the $(m+1)$-dimensional n-order field from the first term of the Eq. (4)

$$\sum_{i\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_i}\right)_{i'} = 0. \tag{5}$$

Then we introduce the generalized canonical momenta for the $(m+1)$-dimensional n-order field

$$p^j = \sum_{i\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_{ji}}\right)_{i'} \quad (j \in \Omega). \tag{6}$$

Obviously,

$$\begin{aligned} p &= \sum_{i\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_i}\right)_{i'}, \\ p^{jx_\theta} &= \sum_{i\in\Omega} (-1)^{\ell} \left(\frac{\partial \mathscr{L}}{\partial u_{jx_\theta i}}\right)_{i'}. \end{aligned} \tag{7}$$

When the boundary conditions are not considered, from the "boundary part" of Eq. (4), we have

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}\left[\sum_{j\in\Omega} p^{jx_\theta}\, du_j\right] = 0. \tag{8}$$

Taking the exterior derivative on both sides of the above equality, we have

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}\left[\sum_{j\in\Omega} dp^{jx_\theta} \wedge du_j\right] = 0. \tag{9}$$

Based on the Eq. (9), we introduce $m+1$ differential two forms

$$\omega_\theta = \sum_{j\in\Omega} dp^{jx_\theta} \wedge du_j \quad (\theta = 0, 1, 2, \ldots, m). \tag{10}$$

So the multisymplectic conservation law for $(m+1)$-dimensional n-order field is obtained as follows:

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta} \omega_\theta = 0. \tag{11}$$

Define the generalized Hamiltonian function for $(m+1)$-dimensional n-order field

$$H = \sum_{\theta=0}^{m} \mathbf{P}^{\mathbf{x}'_\theta} \mathbf{U}_{x_\theta} - \mathscr{L}(u_k) = \sum_{\theta=0}^{m} \sum_{j\in\Omega} p^{jx_\theta} u_{jx_\theta} - \mathscr{L}(u_k) = \sum_{j\in\Omega} p^j u_j - \mathscr{L}(u_k). \tag{12}$$

It is clear that

$$\frac{\partial H}{\partial u_j} = p^j - \frac{\partial \mathscr{L}}{\partial u_j}. \tag{13}$$

Substituting the Euler–Lagrange equation (5) into Eq. (13), we have

$$\frac{\partial H}{\partial u_j} = p^j - \frac{\partial \mathscr{L}}{\partial u_j} = \sum_{i\in\Omega}(-1)^{\ell}\left(\frac{\partial \mathscr{L}}{\partial u_{ji}}\right)_{i'} - \frac{\partial \mathscr{L}}{\partial u_j} = \sum_{i\neq 0}(-1)^{\ell}\left(\frac{\partial \mathscr{L}}{\partial u_{ji}}\right)_{i'} = -\sum_{\theta=0}^{m}\sum_{i\in\Omega}(-1)^{\ell}\left(\frac{\partial \mathscr{L}}{\partial u_{jx_\theta i}}\right)_{i'x_\theta} = -\sum_{\theta=0}^{m}(p^{jx_\theta})_{x_\theta}. \tag{14}$$

The canonical Hamilton equations for $(m+1)$-dimensional n-order field is obtained as follows:

$$\begin{cases} \frac{\partial H}{\partial p^{jx_\theta}} = (u_j)_{x_\theta} \\ \frac{\partial H}{\partial u_j} = -\sum_{\theta=0}^{m}(p^{jx_\theta})_{x_\theta} \end{cases} \quad (j\in\Omega, \theta = 0,1,2,\ldots,m). \tag{15}$$

If we reformulate the canonical Hamilton equations (15) as a matrix Eq. (16), it is just the multisymplectic formulation for $(m+1)$-dimensional n-order field

$$\sum_{\theta=0}^{m} M^{\theta}(Z)_{x_\theta} = \partial H/\partial Z, \tag{16}$$

where

$$Z' = [\mathbf{P}^{\mathbf{x}'_0}, \mathbf{P}^{\mathbf{x}'_1}, \mathbf{P}^{\mathbf{x}'_2}, \ldots, \mathbf{P}^{\mathbf{x}'_\theta}, \ldots, \mathbf{P}^{\mathbf{x}'_m}, \mathbf{U}'],$$

$$\mathbf{P}^{\mathbf{x}_\theta} = \left[p^{\Omega_1 x_\theta}, p^{\Omega_2 x_\theta}, \ldots, p^{\Omega_r x_\theta}, \ldots, p^{\Omega_{\frac{(m+1)^n-1}{m}} x_\theta}\right]',$$

$$\mathbf{U} = \left[u_{\Omega_1}, u_{\Omega_2}, \ldots, u_{\Omega_r}, \ldots, u_{\Omega_{\frac{(m+1)^n-1}{m}}}\right]', \quad (\Omega_r \text{ is the } r\text{th element of } \Omega).$$

M^{θ} is a partitioned matrix, whose elements are $\frac{(m+1)^n-1}{m} \times \frac{(m+1)^n-1}{m}$ submatrices. $M^{\theta}_{\theta+1,m+2} = -M^{\theta}_{m+2,\theta+1} = \mathbf{1}_{\frac{(m+1)^n-1}{m}\times\frac{(m+1)^n-1}{m}}$, the other elements of M^{θ} is $\mathbf{0}_{\frac{(m+1)^n-1}{m}\times\frac{(m+1)^n-1}{m}}$.

3. The multisymplectic conservation law for $(m+1)$-dimensional n-order field

Theorem 1. *There is a natural multisymplectic conservation law in the multisymplectic formulation for $(m+1)$-dimensional n-order field.*

Proof. For $(m+1)$-dimensional n-order field, the variational equation associated with the multisymplectic formulation (16) is

$$\sum_{\theta=0}^{m} M^{\theta}(dZ)_{x_\theta} = D_{ZZ}H\,dZ. \tag{17}$$

Taking the wedge product $\wedge$ on the above Eq. (17) by dZ', where $dZ' = [d\mathbf{P}^{\mathbf{x}'_0}, d\mathbf{P}^{\mathbf{x}'_1}, d\mathbf{P}^{\mathbf{x}'_2}, \ldots, d\mathbf{P}^{\mathbf{x}'_\theta}, \ldots, d\mathbf{P}^{\mathbf{x}'_m}, d\mathbf{U}']$, we have

$$\sum_{\theta=0}^{m}[d\mathbf{P}^{\mathbf{x}'_\theta} \wedge (d\mathbf{U})_{x_\theta} - d\mathbf{U}' \wedge (d\mathbf{P}^{\mathbf{x}_\theta})_{x_\theta}] = dZ' \wedge D_{ZZ}H\,dZ, \tag{18}$$

and

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}(d\mathbf{P}^{\mathbf{x}'_\theta} \wedge d\mathbf{U}) = dZ' \wedge D_{ZZ}H\,dZ. \tag{19}$$

The right side of the Eq. (19) is

$$dZ' \wedge D_{ZZ}H\,dZ = \sum_{\rho,\sigma} dZ_\rho \wedge D_{Z_\rho Z_\sigma}H\,dZ_\sigma = \sum_{\rho,\sigma} D_{Z_\rho Z_\sigma}H\,dZ_\rho \wedge dZ_\sigma. \tag{20}$$

As

$$D_{Z_\rho Z_\sigma}H = D_{Z_\sigma Z_\rho}H, \tag{21}$$

we have

$$D_{Z_\rho Z_\sigma}H\,dZ_\rho \wedge dZ_\sigma + D_{Z_\sigma Z_\rho}H\,dZ_\sigma \wedge dZ_\rho = 0. \tag{22}$$

So

$$dZ' \wedge D_{ZZ}H\,dZ = \sum_{\rho<\sigma} D_{Z_\rho Z_\sigma}H\,dZ_\rho \wedge dZ_\sigma + D_{Z_\sigma Z_\rho}H\,dZ_\sigma \wedge dZ_\rho = 0. \tag{23}$$

From the Eqs. (19) and (23), we have

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}(d\mathbf{P}^{\mathbf{x}'_\theta} \wedge d\mathbf{U}) = 0. \tag{24}$$

Defining $m+1$ differential two forms

$$\omega_\theta = d\mathbf{P}^{\mathbf{x}'_\theta} \wedge d\mathbf{U} = \sum_{\mathrm{i}\in\Omega} dp^{\mathrm{i}x_\theta} \wedge du_{\mathrm{i}}, \tag{25}$$

the multisymplectic conservation law for $(m+1)$-dimensional n-order field is obtained as follows:

$$\sum_{\theta=0}^{m} \frac{\partial}{\partial x_\theta}\omega_\theta = 0. \tag{26}$$

The proof is finished. □

The centred box scheme was used first by Preissman [7] who introduced it for the shallow water equations. It was proved to be a multisymplectic scheme by Brideges and Reich [1]. Hong and Qin [6] generalized the centred box scheme to the case of $(m+1)$-dimensional PDEs. In this paper, we will generalize the centred box scheme to $(m+1)$-dimensional n-order multisimplectic field.

Theorem 2. *The centred box discretization (applying the implicit midpoint rule in the spaces and time respectively) of the multisymplectic formulation (16) for (m + 1)-dimensional n-order field keeps the discrete multisymplectic conservation law.*

Proof. For $(m+1)$-dimensional n-order field, the centred box discretization of the multisymplectic formulation (16) is

$$\sum_{\theta=0}^{m} M^\theta \frac{Z_{I+1/2}^{i_\theta+1} - Z_{I+1/2}^{i_\theta}}{\Delta x_\theta} = \nabla_Z H(Z_{I+1/2}), \tag{27}$$

where

$$\begin{aligned}
&Z_I = Z_{i_0,i_1,i_2,\ldots,i_m},\\
&Z_I^{i_\theta} = Z_{i_0,i_1,i_2,\ldots,i_m},\\
&Z_I^{i_\theta+1} = Z_{i_0,i_1,i_2,\ldots,i_{\theta-1},i_\theta+1,i_{\theta+1},\ldots,i_m},\\
&Z_{I+1/2} = Z_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_m+1/2},\\
&Z_{I+1/2}^{i_\theta} = Z_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_{\theta-1}+1/2,i_\theta,i_{\theta+1}+1/2,\ldots,i_m+1/2},\\
&Z_{I+1/2}^{i_\theta+1} = Z_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_{\theta-1}+1/2,i_\theta+1,i_{\theta+1}+1/2,\ldots,i_m+1/2},\\
&Z_{I+1/2} = \frac{1}{2}(Z_{I+1/2}^{i_\theta} + Z_{I+1/2}^{i_\theta+1}).
\end{aligned} \tag{28}$$

i_θ is the i_θth grid point on the direction of x_θ.

The discrete variational equation associated with (27) is

$$\sum_{\theta=0}^{m} M^\theta \frac{dZ_{I+1/2}^{i_\theta+1} - dZ_{I+1/2}^{i_\theta}}{\Delta x_\theta} = D_{ZZ}H(Z_{I+1/2})dZ_{I+1/2}. \tag{29}$$

Taking the wedge product $\wedge$ on the discrete variational Eq. (29) by

$$dZ'_{I+1/2} = [d\mathbf{P}_{I+1/2}^{\mathbf{x}'_0}, d\mathbf{P}_{I+1/2}^{\mathbf{x}'_1}, d\mathbf{P}_{I+1/2}^{\mathbf{x}'_2}, \ldots, d\mathbf{P}_{I+1/2}^{\mathbf{x}'_\theta}, \ldots, d\mathbf{P}_{I+1/2}^{\mathbf{x}'_m}, d\mathbf{U}'_{I+1/2}],$$

we have

$$\begin{aligned}
&\sum_{\theta=0}^{m}\left[d\mathbf{P}_{I+1/2}^{\mathbf{x}'_\theta} \wedge \frac{d\mathbf{U}_{I+1/2}^{i_\theta+1} - d\mathbf{U}_{I+1/2}^{i_\theta}}{\Delta x_\theta} - d\mathbf{U}'_{I+1/2} \wedge \frac{d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta^{i_\theta+1}} - d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta^{i_\theta}}}{\Delta x_\theta}\right]\\
&\quad = \sum_{\theta=0}^{m}\left[\frac{(d\mathbf{P}_{I+1/2}^{\mathbf{x}'^{i_\theta+1}_\theta} + d\mathbf{P}_{I+1/2}^{\mathbf{x}'^{i_\theta}_\theta}) \wedge (d\mathbf{U}_{I+1/2}^{i_\theta+1} - d\mathbf{U}_{I+1/2}^{i_\theta})}{2\Delta x_\theta} - \frac{(d\mathbf{U}'^{i_\theta+1}_{I+1/2} + d\mathbf{U}'^{i_\theta+1}_{I+1/2}) \wedge (d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta^{i_\theta+1}} - d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta^{i_\theta}})}{2\Delta x_\theta}\right]\\
&\quad = \sum_{\theta=0}^{m}\frac{d\mathbf{P}_{I+1/2}^{\mathbf{x}'^{i_\theta+1}_\theta} \wedge d\mathbf{U}'^{i_\theta+1}_{I+1/2} - d\mathbf{P}_{I+1/2}^{\mathbf{x}'^{i_\theta}_\theta} \wedge d\mathbf{U}_{I+1/2}^{i_\theta}}{\Delta x_\theta} = dZ'_{I+1/2} \wedge D_{ZZ}H(Z_{I+1/2})dZ_{I+1/2} = 0.
\end{aligned} \tag{30}$$

That is

$$\sum_{\theta=0}^{m} \frac{d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta' i_\theta+1} \wedge d\mathbf{U}_{I+1/2}^{\prime i_\theta+1} - d\mathbf{P}_{I+1/2}^{\mathbf{x}_\theta' i_\theta} \wedge d\mathbf{U}_{I+1/2}^{i_\theta}}{\Delta x_\theta} = 0. \tag{31}$$

Defining

$$\omega_{\theta_I} = d\mathbf{P}_I^{\mathbf{x}_\theta'} \wedge d\mathbf{U}_I, \tag{32}$$

where

$$\begin{aligned}
&\omega_{\theta_I} = \omega_{\theta_{i_0,i_1,i_2,\ldots,i_m}},\\
&\omega_{\theta_I}^{i_\theta} = \omega_{\theta_{i_0,i_1,i_2,\ldots,i_m}},\\
&\omega_{\theta_I}^{i_\theta+1} = \omega_{\theta_{i_0,i_1,i_2,\ldots,i_{\theta-1},i_\theta+1,i_{\theta+1},\ldots,i_m}},\\
&\omega_{\theta_{I+1/2}} = \omega_{\theta_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_m+1/2}},\\
&\omega_{\theta_{I+1/2}}^{i_\theta} = \omega_{\theta_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_{\theta-1}+1/2,i_\theta,i_{\theta+1}+1/2,\ldots,i_m+1/2}},\\
&\omega_{\theta_{I+1/2}}^{i_\theta+1} = \omega_{\theta_{i_0+1/2,i_1+1/2,i_2+1/2,\ldots,i_{\theta-1}+1/2,i_\theta+1,i_{\theta+1}+1/2,\ldots,i_m+1/2}},\\
&\omega_{\theta_{I+1/2}} = \frac{1}{2}(\omega_{\theta_{I+1/2}}^{i_\theta+1} + \omega_{\theta_{I+1/2}}^{i_\theta}),
\end{aligned} \tag{33}$$

we obtain the discrete multisymplectic conservation law for $(m+1)$-dimensional n-order field as follows:

$$\sum_{\theta=0}^{m} \frac{\omega_{\theta_{I+1/2}}^{i_\theta+1} - \omega_{\theta_{I+1/2}}^{i_\theta}}{\Delta x_\theta} = 0. \tag{34}$$

The proof is finished. □

4. The local energy and momentum conservation laws for $(m+1)$-dimensional n-order field

In this section, we will show that when $H(Z)$ does not depend explicitly on time or any direction of the space, there is a local energy or momentum conservation law independent of the boundary conditions. Conservation of energy is associated with the translation invariance in time and conservation of momentum is associated with the translation invariance in space.

Theorem 3. *For $(m+1)$-dimensional n-order field, there are natural local energy and momentum conservation laws:*

$$\begin{aligned}
&\frac{\partial E}{\partial x_0} + \sum_{\theta\neq 0} \frac{\partial}{\partial x_\theta} F^\theta = 0,\\
&\frac{\partial G_r}{\partial x_r} + \sum_{\theta\neq r} \frac{\partial}{\partial x_\theta} I_r^\theta = 0,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
&E = \sum_{j\in\Omega} p^{jx_0} u_{jx_0} - \mathscr{L}(u_k),\\
&F^\theta = \sum_{j\in\Omega} p^{jx_\theta} u_{jx_0},\\
&G_r = \sum_{j\in\Omega} p^{jx_r} u_{jx_r} - \mathscr{L}(u_k),\\
&I_r^\theta = \sum_{j\in\Omega} p^{jx_\theta} u_{jx_r}.
\end{aligned} \tag{36}$$

Proof. For $(m+1)$-dimensional n-order field, the multisymplectic formulation is

$$\sum_{\theta=0}^{m} M^\theta Z_{x_\theta} = \nabla_Z H(Z).$$

Taking the right inner product on the above equation by Z_{x_r}, we have

$$\langle \nabla_Z H, Z_{x_r}\rangle = \langle M^r Z_{x_r}, Z_{x_r}\rangle + \sum_{\theta\neq r} \langle M^\theta Z_{x_\theta}, Z_{x_r}\rangle,$$

$$\begin{aligned}\frac{\partial H}{\partial x_r} &= \sum_{\theta\neq r}\langle M^\theta Z_{x_\theta}, Z_{x_r}\rangle = \frac{1}{2}\sum_{\theta\neq r}\langle M^\theta Z_{x_\theta}, Z_{x_r}\rangle + \frac{1}{2}\sum_{\theta\neq r}\langle M^\theta Z_{x_\theta}, Z_{x_r}\rangle \\ &= \frac{1}{2}\sum_{\theta\neq r}\left[\frac{\partial}{\partial x_r}\langle M^\theta Z_{x_\theta}, Z\rangle - \langle M^\theta Z_{x_\theta x_r}, Z\rangle\right] + \frac{1}{2}\sum_{\theta\neq r}\left[\frac{\partial}{\partial x_\theta}\langle M^\theta Z, Z_{x_r}\rangle - \langle M^\theta Z, Z_{x_r x_\theta}\rangle\right] \\ &= \frac{\partial}{\partial x_r}\left[\frac{1}{2}\sum_{\theta\neq r}\langle M^\theta Z_{x_\theta}, Z\rangle\right] - \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}\left[\frac{1}{2}\langle M^\theta Z_{x_r}, Z\rangle\right].\end{aligned} \tag{37}$$

That is

$$\frac{\partial}{\partial x_r}\left[H - \frac{1}{2}\sum_{\theta\neq r}\langle M^\theta Z_{x_\theta}, Z\rangle\right] + \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}\left[\frac{1}{2}\langle M^\theta Z_{x_r}, Z\rangle\right] = 0. \tag{38}$$

From the Eq. (38), we have

$$\begin{aligned}\frac{\partial H}{\partial x_r} - \frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_r}\langle M^\theta Z_{x_\theta}, Z\rangle + \frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}\langle M^\theta Z_{x_r}, Z\rangle &= \frac{\partial H}{\partial x_r} - \frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_r}(\mathbf{U}'_{x_\theta}\mathbf{P}^{\mathbf{x}_\theta} - \mathbf{P}^{\mathbf{x}'_\theta}_{x_\theta}\mathbf{U}) + \frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{U}'_{x_0}\mathbf{P}^{\mathbf{x}_\theta} - \mathbf{P}^{\mathbf{x}'_\theta}_{x_0}\mathbf{U}) \\ &= \left[\frac{\partial H}{\partial x_r} - \frac{1}{2}\frac{\partial}{\partial x_r}\sum_{\theta\neq r}\mathbf{U}'_{x_\theta}\mathbf{P}^{\mathbf{x}_\theta}\right] + \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_\theta x_r}\mathbf{U} + \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_\theta}\mathbf{U}_{x_r} \\ &\quad - \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_r x_\theta}\mathbf{U} - \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_r}\mathbf{U}_{x_\theta} + \left[\frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{U}'_{x_r}\mathbf{P}^{\mathbf{x}_\theta})\right] \\ &= \left[\frac{\partial H}{\partial x_r} - \frac{1}{2}\frac{\partial}{\partial x_r}\sum_{\theta\neq r}\mathbf{U}'_{x_\theta}\mathbf{P}^{\mathbf{x}_\theta}\right] + \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_\theta}\mathbf{U}_{x_r} - \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_r}\mathbf{U}_{x_\theta} \\ &\quad + \left[\frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{U}'_{x_r}\mathbf{P}^{\mathbf{x}_\theta})\right] \\ &= \left[\frac{\partial H}{\partial x_r} - \frac{1}{2}\frac{\partial}{\partial x_r}\sum_{\theta\neq r}\mathbf{U}'_{x_\theta}\mathbf{P}^{\mathbf{x}_\theta}\right] + \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}_{x_\theta}\mathbf{U}_{x_r} - \frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_r}(\mathbf{P}^{\mathbf{x}'_\theta}\mathbf{U}_{x_\theta}) \\ &\quad + \frac{1}{2}\sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}\mathbf{U}_{x_\theta x_r} + \left[\frac{1}{2}\sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{U}'_{x_r}\mathbf{P}^{\mathbf{x}_\theta})\right] \\ &= \frac{\partial}{\partial x_r}\left[H - \sum_{\theta\neq r}\mathbf{P}^{\mathbf{x}'_\theta}\mathbf{U}_{x_\theta}\right] + \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{P}^{\mathbf{x}'_\theta}\mathbf{U}_{x_r}) \\ &= \frac{\partial}{\partial x_r}[\mathbf{P}^{\mathbf{x}'_r}\mathbf{U}_{x_r} - \mathscr{L}(u_k)] + \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}(\mathbf{P}^{\mathbf{x}'_\theta}\mathbf{U}_{x_r}) \\ &= \frac{\partial}{\partial x_r}\left[\sum_{j\in\Omega}p^{jx_r}u_{jx_r} - \mathscr{L}(u_k)\right] + \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}\sum_{j\in\Omega}p^{jx_\theta}u_{jx_r} = 0.\end{aligned}$$

When $r = 0$,

$$\frac{\partial}{\partial x_0}\left[\sum_{j\in\Omega}p^{jx_0}u_{jx_0} - \mathscr{L}(u_k)\right] + \sum_{\theta\neq 0}\frac{\partial}{\partial x_\theta}\sum_{j\in\Omega}p^{jx_\theta}u_{jx_0} = 0. \tag{39}$$

Defining

$$\begin{aligned}E &= \sum_{j\in\Omega}p^{jx_0}u_{jx_0} - \mathscr{L}(u_k), \\ F^\theta &= \sum_{j\in\Omega}p^{jx_\theta}u_{jx_0},\end{aligned} \tag{40}$$

we have

$$\frac{\partial E}{\partial x_0} + \sum_{\theta\neq 0}\frac{\partial F^\theta}{\partial x_\theta} = 0. \tag{41}$$

If $r\neq 0$,

$$\frac{\partial}{\partial x_r}\left[\sum_{j\in\Omega} p^{jx_r}u_{jx_r} - \mathscr{L}(u_k)\right] + \sum_{\theta\neq r}\frac{\partial}{\partial x_\theta}\sum_{j\in\Omega} p^{jx_\theta}u_{jx_r} = 0. \tag{42}$$

Defining

$$\begin{aligned} G_r &= \sum_{j\in\Omega} p^{jx_r}u_{jx_r} - \mathscr{L}(u_k), \\ I_r^\theta &= \sum_{j\in\Omega} p^{jx_\theta}u_{jx_r}, \end{aligned} \tag{43}$$

we have

$$\frac{\partial G_r}{\partial x_r} + \sum_{\theta\neq r}\frac{\partial I_r^\theta}{\partial x_\theta} = 0. \tag{44}$$

The proof is finished. □

5. The multisymplectic formulation for the KP equation

The Kadomtsev–Petviashvili equation [8] is

$$(2u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0. \tag{45}$$

To place the KP equation in the variational framework, we let $u = \varphi_{xx}$, then φ satisfies equation:

$$2\varphi_{xxxt} + 6\varphi_{xx}\varphi_{xxxx} + 6\varphi_{xxx}^2 + \varphi_{xxxxxx} + \sigma\varphi_{xxyy} = 0, \tag{46}$$

and

$$\varphi_{xtxx} + \varphi_{xxtx} + 6\varphi_{xx}\varphi_{xxxx} + 6\varphi_{xxx}^2 + \varphi_{xxxxxx} + \sigma\varphi_{xyyx} = 0. \tag{47}$$

The corresponding Lagrangian density can be written as

$$\mathscr{L} = \varphi_{xx}\varphi_{xt} - \frac{1}{2}\varphi_{xxx}^2 + \frac{\sigma}{2}\varphi_{xy}^2 + \varphi_{xx}^3. \tag{48}$$

Obviously, the field is three order, so the Ω for (2 + 1)-dimensional third-order field is

$$\left\{\begin{matrix} 0, t, x, y, tt, tx, ty, xt, xx, xy, yt, yx, yy, \\ ttt, ttx, tty, txt, txx, txy, tyt, tyx, tyy, xtt, xtx, xty, xxt, xxx, xxy, xyt, xyx, xyy, ytt, ytx, yty, yxt, yxx, yxy, yyt, yyx, yyy \end{matrix}\right\}. \tag{49}$$

Using the Eq. (6), we have

$$\begin{aligned} p^{xt} &= \varphi_{xx}, \\ p^{x} &= -\varphi_{xxt} - \varphi_{xtx} - 6\varphi_{xx}\varphi_{xxx} - \sigma\varphi_{xyy} - \varphi_{xxxxx}, \\ p^{xx} &= \varphi_{xt} + 3\varphi_{xx}^2 + \varphi_{xxxx}, \\ p^{xxx} &= -\varphi_{xxx}, \\ p^{xy} &= \sigma\varphi_{xy}, \\ p^{t} &= p^{tt} = p^{yt} = p^{ttt} = p^{txt} = p^{tyt} = p^{xtt} = p^{xxt} = p^{xyt} = p^{ytt} = p^{yxt} = p^{yyt} \\ &= p^{tx} = p^{yx} = p^{ttx} = p^{txx} = p^{tyx} = p^{xtx} = p^{xyx} = p^{ytx} = p^{yxx} = p^{yyx} = p^{y} \\ &= p^{ty} = p^{yy} = p^{tty} = p^{txy} = p^{tyy} = p^{xty} = p^{xyy} = p^{yty} = p^{yxy} = p^{yyy} \\ &= 0. \end{aligned} \tag{50}$$

The Hamiltonian function and the energy density for the KP equation (reference (12) and (40)) are respectively

$$H = p^{xt}\varphi_{xt} + p^{x}\varphi_{x} + p^{xx}\varphi_{xx} + p^{xxx}\varphi_{xxx} + p^{xy}\varphi_{xy} - \mathscr{L}(\varphi_{xt},\varphi_{xx},\varphi_{xy},\varphi_{xxx}) = p^{x}\varphi_{x} + p^{xx}\varphi_{xx} - \frac{p^{xxx^2}}{2} + \frac{p^{xy^2}}{2\sigma} - \varphi_{xx}^3, \tag{51}$$

$$E = \sum_{j\in\Omega} p^{jx_0}u_{jx_0} - \mathscr{L}(u_k) = p^{xt}\varphi_{xt} - \mathscr{L}(u_k) = -\frac{p^{xxx^2}}{2} + \frac{p^{xy^2}}{2\sigma} + \varphi_{xx}^3. \tag{52}$$

By the Eq. (16), the multisymplectic formulation for (2 + 1)-dimensional third-order field can be written as

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P^t} \\ \mathbf{P^x} \\ \mathbf{P^y} \\ \mathbf{U} \end{bmatrix}_t + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P^t} \\ \mathbf{P^x} \\ \mathbf{P^y} \\ \mathbf{U} \end{bmatrix}_x + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P^t} \\ \mathbf{P^x} \\ \mathbf{P^y} \\ \mathbf{U} \end{bmatrix}_y = \begin{bmatrix} \partial\mathbf{H}/\partial\mathbf{P^t} \\ \partial\mathbf{H}/\partial\mathbf{P^x} \\ \partial\mathbf{H}/\partial\mathbf{P^y} \\ \partial\mathbf{H}/\partial\mathbf{U} \end{bmatrix}. \tag{53}$$

$$\begin{aligned} \mathbf{P^t} &= [p^t, p^{tt}, p^{xt}, p^{yt}, p^{ttt}, p^{txt}, p^{tyt}, p^{xtt}, p^{xxt}, p^{xyt}, p^{ytt}, p^{yxt}, p^{yyt}]', \\ \mathbf{P^x} &= [p^x, p^{tx}, p^{xx}, p^{yx}, p^{ttx}, p^{txx}, p^{tyx}, p^{xtx}, p^{xxx}, p^{xyx}, p^{ytx}, p^{yxx}, p^{yyx}]', \\ \mathbf{P^y} &= [p^y, p^{ty}, p^{xy}, p^{yy}, p^{tty}, p^{txy}, p^{tyy}, p^{xty}, p^{xxy}, p^{xyy}, p^{yty}, p^{yxy}, p^{yyy}]', \\ \mathbf{U} &= [u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xt}, u_{xx}, u_{xy}, u_{yt}, u_{yx}, u_{yy}]', \end{aligned}$$

1 is a 13×13 identity matrix, and **0** is a 13×13 zero matrix.

Substituting (50) into (53), the multisymplectic formulation can be simplified as

$$\begin{bmatrix} 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} p^{xt} \\ p^x \\ p^{xx} \\ p^{xxx} \\ p^{xy} \\ \varphi \\ \varphi_x \\ \varphi_{xx} \end{bmatrix}_t + \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0 \\ 0&0&-1&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0 \end{bmatrix} \begin{bmatrix} p^{xt} \\ p^x \\ p^{xx} \\ p^{xxx} \\ p^{xy} \\ \varphi \\ \varphi_x \\ \varphi_{xx} \end{bmatrix}_x$$

$$+ \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} p^{xt} \\ p^x \\ p^{xx} \\ p^{xxx} \\ p^{xy} \\ \varphi \\ \varphi_x \\ \varphi_{xx} \end{bmatrix}_y = \begin{bmatrix} \partial H/p^{xt} \\ \partial H/p^x \\ \partial H/p^{xx} \\ \partial H/p^{xxx} \\ \partial H/p^{xy} \\ \partial H/\varphi \\ \partial H/\varphi_x \\ \partial H/\varphi_{xx} \end{bmatrix}. \tag{54}$$

Eq. (54) is the multisymplectic formulation for the KP equation.

6. Conclusions and future work

In this paper, we establish a multisymplectic framework for $(m+1)$-dimensional n-order field. Based on the multisymplectic formulation, we directly derive the multisymplectic conservation law and the local energy and momentum conservation laws for $(m+1)$-dimensional n-order field. Then we prove that the centred box discretization of the multisymplectic formulation for $(m+1)$-dimensional n-order field is a multisymplectic scheme. As an example, the process of constructing the multisymplectic formulation for the KP equation is given.

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