Fidelity of Quantum Teleportation for Single-Mode Squeezed State Light *

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The fidelity of quantum teleportation of a single-mode squeezed state of light is calculated based on the general theory of quantum-mechanical measurement in the Schrödinger picture. It is shown that the criterion for the nonclassical state teleportation is different from that for coherent state. $F = 1/2$ is no longer the rigorous boundary between classical and quantum teleportation for a squeezed state of light. When the quantum entanglement of an Einstein–Podolsky–Rosen (EPR) beam used for teleportation and the parameters of the system are given, the fidelity depends on the squeezing of the input squeezed state. The higher the squeezing is, the smaller the fidelity is, and the lower the classical limitation of fidelity is. The dependence of the optimum gain for teleporting a squeezed vacuum state upon the EPR entanglement is also calculated. The results obtained provide important references for designing experimental systems of teleporting a non-classical state and judging the quality of the teleported quantum state.

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Quantum teleportation is a method by which quantum information encoded in a quantum state can be successfully retrieved at a distant location. It represents the basic building block of future quantum communication networks between parties and has potential applications in quantum computing[1] and quantum information manipulation.[2]

Quantum teleportation was firstly proposed for single particles,[3] the quantum teleportation of single photon polarization states was also experimentally demonstrated.[4,5] Recently, it has been extended to the coherent state of continuous electromagnetic fields[6,7] and has been demonstrated.[8] For an ideal teleportation, the output state emerging at the receiving station should perfectly mimics the original unknown input state. However for any real situation, the Einstein–Podolsky–Rosen (EPR) entanglement used as quantum channel of teleportation is imperfect, because the perfect entanglement requires the perfect squeezing which corresponds to an infinite amount of energy of the light field, that is, the perfect teleportation is impossible to be obtained in experiments, the fidelity is used to evaluate the success of the quantum teleportation process. Recently various discussions on this subject have appeared.[9−11] Up to now, most work has focused on the case of coherent input states which are very close to classical states. When nonclassical states of light are used for communication, the channel capacity and the signal-to-noise ratio can be improved,[12] thus it is important to pay attention to the teleportation of original nonclassical states.[13] The discussions on the entanglement swapping for discrete variables[14−18] and continuous electromagnetic field[19−21] firstly enter into the investigation for teleporting nonclassical features of quantum states. The transfer of nonclassical features in quantum teleportation of nonclassical states, such as the squeezed vacuum and Fock state has been studied in the case that the quantum channel is influenced by a thermal environment and the normalized classical gain of the system is designated to the maximum value of 1.[22] In fact, the unit gain can only be used for maximizing the fidelity for coherent-state teleportation with an amplitude much larger than one.[23] The nonclassical properties of the teleported quantum state which originally has nonclassical properties were theoretically discussed.[24] Very recently, the squeezed-state teleportation was reported and the corresponding fidelity for the squeezed thermal state was given to evaluate the efficiency of teleportation of nonclassical state.[25]

In this Letter, we calculate the fidelity of teleportation for squeezed states based on the general theory of quantum-mechanic measurement in the Schrödinger picture. The obtained results show that the fidelity and the classical boundary not only depend on the EPR entanglement and also the squeezing of the input squeezed state. The criteria differences between the teleportation of coherent states and squeezed states are discussed. Our calculations indicate that it is not appropriate applying the criteria derived from the coherent state to the teleportation of squeezed states. It has been theoretically demonstrated that the higher EPR entanglement has to be required for teleporting a squeezed state with the same fidelity with respect to a coherent state. We also calculate the optimum gain for teleporting a squeezed vacuum state. Our results provide useful references for designing the system teleporting squeezed states and better standards for estimating the quality of finished teleportation for squeezed states.

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The EPR entangled state used in the proposed teleportation scheme for continuous variables is a two-mode squeezed vacuum state which can be expanded in a Fock state basis,[26]

$$\rho_{1,2} = (1-\lambda^2)\sum_{n}^{\infty}\sum_{n'}^{\infty}(-\lambda)^{n+n'}|n,n\rangle_{1,2}\langle n',n'|, \quad (1)$$

where the indices 1 and 2 denote the two modes of the squeezed state; λ $(0<\lambda<1)$ indicates the degree of EPR entanglement; $\lambda=0$ represents the classical limit without any EPR entanglement. At the limit of infinite squeezing $\lambda\to 1$, the two-mode squeezed state approaches a maximally entangled state for quadrature phase amplitudes X and Y,[27,28] thus it is the analogue of the ideal EPR beam with perfect entanglements between quadrature phase amplitudes.

For the teleportation of an arbitrary pure state $|\Psi\rangle_{\rm in}$, the density operator of the unknown input state takes the form

$$\hat{\rho}_{\rm in} = |\Psi\rangle_{\rm in\ in}\langle\Psi|. \quad (2)$$

Using the same discussion[29] as described in Ref. [29], the input state after experiencing all of the processes of teleportation is transformed into

$$\hat{\rho}_2^{\rm out} = \big[\sqrt{2/\pi}\sqrt{1-\lambda^2}e^{-(X^2+Y^2)}\hat{D} \cdot(\sqrt{2}g(X+iY))|\Phi\rangle\otimes{\rm H.C.}\big]\cdot\big[(2/\pi)(1-\lambda^2) \cdot e^{-2(X^2+Y^2)}f(\gamma,m)f^*(\gamma',m)\big]^{-1}, \quad (3)$$

where

$$|\Phi\rangle = f(\gamma,m)\hat{D}(\lambda[\gamma-\sqrt{2}(X+iY)])|0\rangle_2,$$

$$f(\gamma,m) = \frac{1}{\pi}\int d^2\gamma e^{-\frac{1}{2}|\gamma|^2} \cdot e^{\sqrt{2}(X-iY)}e^{\frac{1}{2}\lambda^2|\gamma-\sqrt{2}(X+iY)|^2}\langle\gamma|\Psi\rangle_{\rm in},$$

and the displacement operator

$$\hat{D}(\sqrt{2}g(X+iY)]) = e^{-g^2(X+iY)|^2}e^{-\sqrt{2}g(X+iY)]\hat{a}_2^+} \cdot e^{-\sqrt{2}g(X+iY)]\hat{a}_2},$$

where g represents a normalized classical gain for the transformation from classical measured infinitesimal values X and Y to complex field amplitudes $X+iY$.

In the limit of infinite squeezing $\lambda=1$ and unity classical transformation gain $g=1$, we can obtain the density matrix of the conditional output state of the other half of EPR pair:

$$\hat{\rho}_2^{\rm out} = \hat{\rho}_{\rm in}. \quad (4)$$

It is clear that under this limit the input unknown pure quantum state can be successfully teleported to the receiving station and can be perfectly retrieved by the other half of the EPR beam.

In fact the maximally entangled squeezed state is unphysical since it needs an infinite amount of energy of the light field. For a real teleportation system with finite entanglement the input and output states are not exactly the same and only partly overlapped. Considering the real case of $\lambda<1$ and $g\neq 1$, the output state expressed by Eq. (3) behaves like a mixture of the unnormalized density matrix elements. The average density matrix of output state is

$$\hat{\rho}_T = \sqrt{(2/\pi)(1-\lambda^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dXdYe^{-(X^2+Y^2)} \cdot\hat{D}(\sqrt{2}g(X+iY))|\Phi\rangle\otimes{\rm H.c.} \quad (5)$$

The fidelity for evaluating the efficiency of realistic- and thus imperfect- teleportation of quantum states is given from the initial definition:[30]

$$F =_{\rm in}\langle\Psi|\hat{\rho}_T|\Psi\rangle_{\rm in}. \quad (6)$$

Substituting Eq.(5) into Eq. (6) the fidelity is written in the following form:

$$F = (1-\lambda^2)\frac{2}{\pi}\Big(\frac{1}{\pi}\Big)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dXdY \cdot e^{-2(1-\lambda^2)(X^2+Y^2)}\int d^2\beta e^{-\frac{1}{2}(1-\lambda^2)|\beta|^2} \cdot e^{\beta(X-iY)\left(\sqrt{2}-\frac{\lambda(g+\lambda)}{\sqrt{2}}\right)}e^{\beta^*(X+iY)\frac{\lambda(g-\lambda)}{\sqrt{2}}} \cdot\langle\beta|\Psi\rangle_{\rm in}\langle\Psi|\lambda\beta+\sqrt{2}(g-\lambda)(X+iY)\rangle \cdot\int d^2\gamma e^{-\frac{1}{2}(1-\lambda^2)|\gamma|^2}e^{\gamma^*(X+iY)(\sqrt{2}-\frac{\lambda(g+\lambda)}{\sqrt{2}})} \cdot e^{\gamma(X-iY)\frac{\lambda(g-\lambda)}{\sqrt{2}}}{}_{\rm in}\langle\Psi|\gamma\rangle\langle\lambda\gamma+\sqrt{2}(g-\lambda) \cdot(X+iY)|\Psi\rangle_{\rm in}, \quad (7)$$

where β and γ are the coherent amplitudes of coherent state bases $|\beta\rangle$ and $|\gamma\rangle$.

For an input single-mode squeezed state of light the wavefunction is expressed by

$$|\Psi\rangle_{\rm in} = |\alpha,\zeta\rangle = \hat{S}(\zeta)|\alpha\rangle, \quad (8)$$

where $\hat{S}(\zeta)=e^{(-\frac{1}{2}\zeta\hat{a}_{\rm in}^{+2}+\frac{1}{2}\zeta^*\hat{a}_{\rm in}^2)}$ is the squeezing operator, $\zeta=r_0e^{i\varphi_0}$ is the squeezing parameter with modulus r_0 and argument φ_0. The exponential function of operator $\hat{S}(\zeta)$ is derived from the operator ordering theorem:

$$\hat{S}(\zeta) = e^{-\frac{1}{2}\hat{a}_{\rm in}^{+2}\exp(i\varphi_0)\tanh r_0}e^{-\frac{1}{2}(1+2\hat{a}_{\rm in}^+\hat{a}_{\rm in})\ln(\cosh r_0)} \cdot e^{\frac{1}{2}\hat{a}_{\rm in}^2\exp(i\varphi_0)\tanh r_0}, \quad (9)$$

setting $\varphi_0=0$ for simplicity and without the loss of generality, Eq. (8) is expanded to

$$|\alpha,\zeta\rangle = \frac{1}{\sqrt{\cosh r_0}}e^{\frac{1}{2}(\frac{1}{|\cosh r_0|^2}-1)|\alpha|^2} \cdot e^{\frac{1}{2}\alpha^2\tanh r_0}e^{-\frac{1}{2}\hat{a}_{\rm in}^{+2}\tanh r_0}|\frac{\alpha}{\cosh r_0}\rangle, \quad (10)$$

and we have

$$\langle\beta|\alpha,\zeta\rangle = \frac{1}{\sqrt{\cosh r_0}}e^{-\frac{1}{2}|\beta|^2}e^{-\frac{1}{2}|\alpha|^2} \cdot e^{\frac{1}{2}\alpha^2\tanh r_0}e^{-\frac{1}{2}\beta^{*2}\tanh r_0}e^{-\frac{\beta^*\alpha}{\cosh r_0}}. \quad (11)$$

The average fidelity for an input single-mode squeezed state is obtained by integrating out the parameters β, γ, X and Y in Eq. (7),

$$
\begin{aligned}
F = &\frac{(1-\lambda^2)}{\cosh^2 r_0(1-\lambda^2\tanh^2 r_0)\sqrt{AB}} \\
&\cdot \exp\Big\{-2\Big(1-\frac{\lambda}{\cosh^2 r_0(1-\lambda^2\tanh^2 r_0)}\Big)|\alpha|^2\Big\} \\
&\cdot \exp\Big\{\Big(\tanh r_0 - \frac{\lambda^2\tanh r_0}{\cosh^2 r_0(1-\lambda^2\tanh^2 r_0)}\Big) \\
&\cdot (\alpha^2+\alpha^{*2})\Big\} \\
&\cdot \exp\Big\{\frac{(1-\lambda)^2(1+g)^2(1-\lambda\tanh r_0)}{4A\cosh^2 r_0(1-\lambda^2\tanh^2 r_0)^2}(\alpha+\alpha^*)^2\Big\} \\
&\cdot \exp\Big\{-\frac{(1-\lambda)^2(1+g)^2(1+\lambda\tanh r_0)}{4B\cosh^2 r_0(1-\lambda^2\tanh^2 r_0)^2} \\
&\cdot (\alpha^*-\alpha)^2\Big\}, \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
A = &1-\lambda^2+(g-\lambda)^2(1+\tanh r_0) \\
&+\frac{\tanh r_0[(1-g\lambda)-\lambda(g-\lambda)\tanh r_0]^2}{1-\lambda^2\tanh^2 r_0}, \\
B = &1-\lambda^2+(g-\lambda)^2(1-\tanh r_0) \\
&+\frac{\tanh r_0[(1-g\lambda)+\lambda(g-\lambda)\tanh r_0]^2}{1-\lambda^2\tanh^2 r_0}.
\end{aligned}
$$

If we take $r_0 = 0$, i.e., when a coherent state is teleported, Eq.(12) becomes

$$
F = \frac{1-\lambda^2}{1+g^2-2g\lambda} e^{-(1-g)^2\frac{1-\lambda^2}{1+g^2-2g\lambda}|\alpha|^2}, \qquad (13)
$$

which is the same with the fidelity for coherent state teleportation given in Ref. [23].

In the case of large coherent amplitude ($|\alpha|^2 \gg 0$), it is found from Eq. (12) that the maximum fidelity is obtained at the unit gain $g = 1$ independent of squeezing r_0 and EPR entanglement λ. Therefore for the teleportation of single-mode squeezed coherent state with large coherent amplitude, the classical gain of the unit should always be taken and the maximum fidelity is given

$$
\begin{aligned}
F = &(1+\lambda)(2\cosh^2 r_0)^{-1}[(1-\lambda^2\tanh^2 r_0) \\
&\cdot(1+\tanh r_0)(1-\lambda\tanh r_0 \\
&+\lambda(1-\lambda)\tanh^2 r_0)]^{-1/2}. \qquad (14)
\end{aligned}
$$

Fig. 1 shows the dependence of the fidelity given in Eq. (14) on the modulus of the squeezing parameter of the teleported squeezed state with respect to the designated parameters (λ) of the EPR entanglement. For the ideal case of $\lambda \to 1$ and $g = 1$, the quantum fidelity equals to 1 (curve e in Fig. 1) and it no longer depends on the squeezing of the input state as mentioned above. However, for imperfect entanglement ($0 < \lambda < 1$), the fidelity drops when the squeezing (r_0) of input squeezed state increases for a given EPR entanglement [curves $d-a$, $\lambda = 0.9$, 0.6, 0.3, 0]. The results tell us that the single-mode squeezed state with higher squeezing is more difficult to be teleported with high fidelity. Compared to the teleportation of coherent states the higher entanglement of EPR is required. That is because the quantum information included in the squeezed state with higher squeezing is more than that with lower squeezing, thus the high nonlocal entanglement has to be used for teleporting the more quantum information. The curve a in Fig. 1 represents the case without entanglement ($\lambda = 0$), which indicates the boundary F_{class} between the classical and quantum domains for the teleportation of squeezed coherent state.

$$
F_{\text{class}} = \frac{1}{2\cosh^2 r_0\sqrt{(1+\tanh r_0)}}. \qquad (15)
$$

It is obvious that both the fidelity and the boundary for the teleportation of the single-mode squeezed coherent state of light are dependent on the squeezing r_0 of the input unknown squeezed state. For a given entanglement λ, the higher the squeezing is, the smaller the fidelity is and the lower the boundary is. Only when $r_0 = 0$, the fidelity of curve a equals 1/2 which is just the classical boundary obtained for the coherent input states.[9]

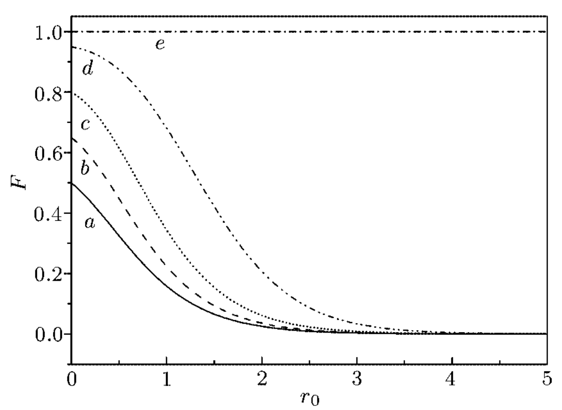

Fig. 1. The fidelity F versus the squeezing of the input squeezed state for different parameters of quantum correlation between EPR pair; curve a, $\lambda = 0$, i.e. classical limit for EPR entanglement; curve b, $\lambda = 0.3$; curve c, $\lambda = 0.6$; curve d, $\lambda = 0.9$; curve e, $\lambda = 1$, i.e. ideal EPR pair.

In the above analysis, we consider the teleportation of squeezed coherent state ($\alpha \neq 0$) in which the unit gain is always corresponding to the optimal fidelity. However, for the teleportation of a squeezed vacuum state ($\alpha = 0$), our calculation shows that the maximal fidelity is obtained at $g < 1$. Taking $\alpha = 0$ we numerically solved Eq. (12) and obtained the function of the fidelity versus g at $r_0 = 0.3$ and $\lambda = 0$, 0.3, 0.6 and 0.8 [curves $a-d$ in Fig. 2]. The peaks of the curves appear at the gain of $g < 1$, this means that the maximum fidelity corresponds to $g \neq 1$ and when λ also increases the optimal gain increases. Therefore in implementing teleportation experiment the receiver should adjust the classical gain of the system carefully to reach the optimal fidelity for each given EPR source.

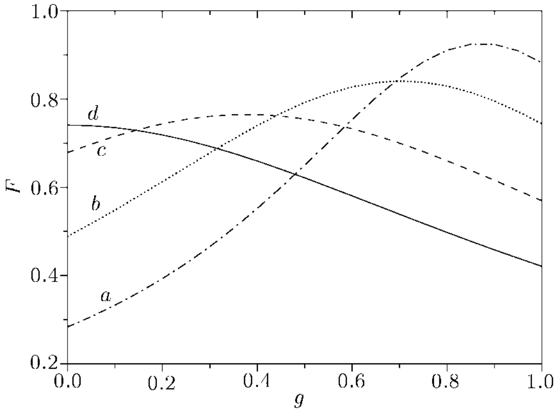

Fig. 2. The fidelity F versus the classical gain with the input squeezed vacuum state with the squeezing $r_0 = 0.3$ and the different entanglement of EPR. Curve a, $\lambda = 0$; Curve b, $\lambda = 0.3$; curve c, $\lambda = 0.6$; curve d, $\lambda = 0.8$.

The fidelity and classical boundary for the quantum teleportation of squeezed states have been analytically calculated based on the theory of quantum-mechanical measurement in the Schrödinger picture and the initial definition of fidelity. The obtained results indicate that the fidelities and the classical boundaries of the squeezed state teleportation and the coherent state teleportation are different. Therefore, it is not appropriate to apply the criteria derived from the coherent state in the teleportation of the squeezed state. Figure 1 shows that the fidelity and the classical boundary of a teleported squeezed state depend on the squeezing of the input state. Although the teleported state is an unknown state for the sender, when the receiver has retrieved the teleported state he should know what it is. If a squeezed state is retrieved, the receiver should not evaluate the quality of the teleportation using the criteria for the coherent state and should use that for the squeezed state. For any real and thus imperfect teleportation the retrieved quantum state is not perfectly the same with the input state, and the squeezing of the retrieved state would be less than that of the original state. However, even so, estimating the quality of the teleportation of a squeezed state in terms of the squeezing measured by the receiver and the criteria derived specifically for the squeezed state teleportation should be more reasonable and closer to the real situation than using that obtained for the coherent state. In fact, for the teleportation, an unknown quantum one even does not know that it is a coherent state or anything else before it is retrieved, so the evaluating criteria only can be selected after retrieved. Our calculations prove that using a given EPR entanglement the fidelity and its classical limit of teleporting a squeezed state is smaller than that teleporting a coherent state. In the viewpoint of physics, the result is reasonable because the quantum information included in the squeezed state is more than that in the coherent state, thus the higher nonlocal entanglement should be required for teleporting the more quantum information. When one designs a teleportation system and intends to teleport the squeezed state, the present results may provide useful reference for preparing an available EPR source. In addition, we resolve the optimal classical gain of teleporting a squeezed state and point out that when a squeezed vacuum state is teleported, the optimal gain is dependent on the used EPR entanglement and not equals to 1. Therefore, in performing the teleportation experiments the receiver should adjust the classical gain of the system carefully to reach the best fidelity for each given EPR source.

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