Generation of qudits and entangled qudits

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We propose schemes for the generation of d-level quantum states (qudits) and quantum entangled states of N qudits by adopting the polarization degree of freedom of photons. Arbitrary qudits can be prepared by putting $d-1$ identical and independent photons (each with a given polarization) into a single spatial and temporal mode. Then, by using N such qudits, linear optics, and proper projection measurements, quantum entangled states of N qudits can be produced. The frequencies of the N qudits can be identical or different according to the corresponding linear optics and projection measurements.

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The quantum-mechanical analog of the classical bit is the qubit, a two-level quantum system. It is known that qubits and entangled qubits play a key role in quantum-information processing. Qudits and entangled qudits, a natural extension of two-level quantum systems to d-level quantum systems, have raised considerable interest recently. For fundamental aspects of quantum theory, it is shown that violations of local realism by two entangled d-dimensional systems are stronger than those by two qubits [1]. In quantum key distribution, protocols with qutrits ($d=3$) and ququads ($d=4$) instead of qubits have been demonstrated to be more resilient to a specific class of eavesdropping attacks [2–5]. In addition, the higher-dimensional Hilbert space associated with these states may allow the realization of new types of quantum protocol.

In the past few years, some optical realizations and applications of qudits and entangled qudits using various degrees of freedom have been demonstrated. Generation of an entangled qutrit state has been proposed and performed using orbital angular momentum [6,7]. By using the transverse momentum and position entanglement of photons, pixel entanglement could be produced [8,9]. Energy-time entangled qutrits and time-bin entanglement up to $d=20$ have been reported recently [10,11]. The experimental realization of qutrit states and qudit states by adopting the polarization degree of freedom of two-photon state was also reported [12–16]. All the experiments mentioned above are based on spontaneous parametric down-conversion (SPDC); however, the nonlinear interactions are not necessary. Recently, it has been shown that multiphoton entanglement can be created and manipulated using only single-photon sources, linear optics, and precise photon detection [17–20]. In this paper, by using such linear optics method, we propose schemes for generation of qudits and N entangled qudits by adopting the polarization degree of freedom of photons.

The proposed schematic setup for the generation of arbitrary qudits is shown in Fig. 1 [19]. For simplicity, we first show that how an arbitrary qutrit can be created. An arbitrary qutrit can be written as

$$|\Psi_3\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \quad (1)$$

where c_i $(i=1,2,3)$ are complex probability amplitudes and satisfy $|c_1|^2+|c_2|^2+|c_3|^2=1$. By using the definitions

$$|1\rangle \to |2H,0V\rangle, \quad |2\rangle \to |1H,1V\rangle, \quad |3\rangle \to |0H,2V\rangle, \quad (2)$$

where, for example, $|2H,0V\rangle$ means a single-mode quantum state with two horizontal polarization photons and zero vertical polarization photons, Eq. (1) can be rewritten in terms of creation operators,

$$|\Psi_3\rangle = \left(\frac{1}{\sqrt{2}}c_1(\hat{a}_H^\dagger)^2 + c_2\hat{a}_H^\dagger\hat{a}_V^\dagger + \frac{1}{\sqrt{2}}c_3(\hat{a}_V^\dagger)^2\right)|\mathrm{vac}\rangle. \quad (3)$$

To prepare the above state, two single-photon Fock states with a certain polarization are injected into two input ports of a nonpolarizing beam splitter. The input state can be written as

$$|\Psi_{\mathrm{in}}\rangle = \prod_{l=1}^{2} (\cos\theta_l \hat{a}_{l,H}^\dagger - \sin\theta_l e^{i\phi_l}\hat{a}_{l,V}^\dagger)|\mathrm{vac}\rangle, \quad (4)$$

where $\hat{a}_{l,H}^\dagger$ and $\hat{a}_{l,V}^\dagger$ are the photon creation operators of horizontal and vertical polarization in the spatial mode l. $\cos\theta_l$, $\sin\theta_l$, and $e^{i\phi_l}$ define the polarization states of the single-photon states. To transfer the two input modes into a single-mode output, a photon detector in the unused output port of the beam splitter is employed to detect a vacuum state (no photon is detected). After this postselection procedure, the output two-photon state reads

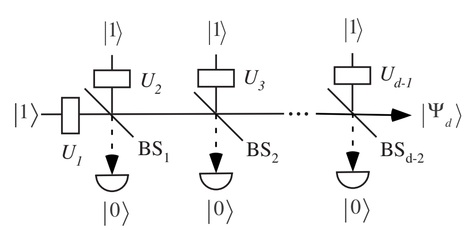

FIG. 1. Schematic setup for the generation of arbitrary qudits. BS_1, BS_2,..., BS_{d-2} are nonpolarizing beamsplitters. U_1, U_2,..., U_{d-1} denote the unitary operations that can adjust the polarization states of the single-photon Fock states arbitrarily.

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$$|\Psi_{\text{out}}\rangle = p_2 \prod_{l=1}^{2} (\cos\,\theta_l \hat{a}_H^\dagger - \sin\,\theta_l e^{i\phi_l} \hat{a}_V^\dagger)|\text{vac}\rangle, \tag{5}$$

where p_2 is the postselection probability amplitude. If we set $\tan\,\theta_l e^{i\phi_l} = x_l$ [here x_l $(l=1,2)$ are complex roots of the equation $\frac{1}{\sqrt{2}}c_1 x^2 + c_2 x + \frac{1}{\sqrt{2}}c_3 = 0$], then the quantum state described by Eq. (5) and that described by Eq. (3) will be completely identical. For instance, if we want to prepare a qutrit with $c_1 = c_2 = c_3 = 1/\sqrt{3}$, the polarization states of the two single-photon state should be set as $(\sqrt{2}/2)(\hat{a}_H^\dagger + e^{i\pi/4}\hat{a}_V^\dagger)|\text{vac}\rangle$ and $(\sqrt{2}/2)(\hat{a}_H^\dagger + e^{-i\pi/4}\hat{a}_V^\dagger)|\text{vac}\rangle$.

The above mentioned method can be directly generalized to prepare arbitrary qudits

$$|\Psi_d\rangle = \sum_{j=1}^{d} c_j |j\rangle, \tag{6}$$

where $\Sigma_{j=1}^{d}|c_j|^2 = 1$, and the correspondence between $|j\rangle$ and the single-mode multiphoton polarization states is $|j\rangle = |(d-j)H,(j-1)V\rangle$. In terms of photon creation operators, Eq. (6) can be rewritten as

$$|\Psi_d\rangle = \sum_{j=1}^{d} \frac{c_j}{\sqrt{(d-j)!(j-1)!}} (\hat{a}_H^\dagger)^{d-j} (\hat{a}_V^\dagger)^{j-1} |\text{vac}\rangle. \tag{7}$$

To produce the quantum state $|\Psi_d\rangle$, $d-1$ single-photon Fock states with given polarizations are injected into the input ports of a series of nonpolarizing beam splitters (Fig. 1), the input quantum states is given by

$$|\Psi_{\text{in}}\rangle = \prod_{l=1}^{d-1} (\cos\,\theta_l \hat{a}_{l,H}^\dagger - \sin\,\theta_l e^{i\phi_l} \hat{a}_{l,V}^\dagger)|\text{vac}\rangle. \tag{8}$$

We consider only those events where all photon detectors in the unused output ports of the beam splitters do not detect any photons. Then, all the input single-photon Fock states are transferred to a single-mode output. The postselected output multiphoton state can be expressed as

$$|\Psi_{\text{out}}\rangle = p_d \prod_{l=1}^{d-1} (\cos\,\theta_l \hat{a}_H^\dagger - \sin\,\theta_l e^{i\phi_l} \hat{a}_V^\dagger)|\text{vac}\rangle, \tag{9}$$

where p_d is the postselection probability amplitude. By setting $\tan\,\theta_l e^{i\phi_l} = x_l$, where x_l $(l=1,2,\ldots,d-1)$ are complex roots of the equation

$$\sum_{j=1}^{d} \frac{c_j}{\sqrt{(d-j)!(j-1)!}} x^{d-j} = 0, \tag{10}$$

arbitrary high-dimensional quantum states (6) for any d can be prepared. The validity of the above mentioned results comes from the main theorem of polynomial algebra: every polynomial equation of degree d with complex coefficients has d roots in the complex numbers.

In the following, we show that, starting with two qudits (6), two entangled qudits can be conditionally generated via the projection measurement (see Fig. 2). First, two qudits are injected into two input ports of a polarizing beam splitter (PBS). The input state can be given by

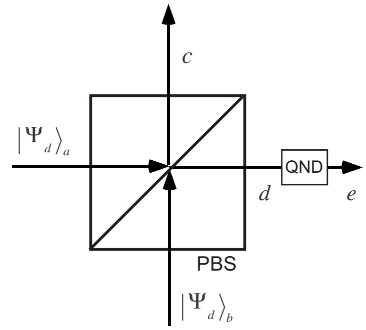

FIG. 2. Schematic setup for the generation of two entangled qudits. PBS, polarizing beam splitter; QND, quantum nondemolition measurements.

$$|\Psi_{\text{in}}\rangle = |\Psi_d\rangle_a \otimes |\Psi_d\rangle_b = \sum_{j=1}^{d}\sum_{k=1}^{d} c_j e_k |j\rangle_a |k\rangle_b, \tag{11}$$

where $\Sigma_{j=1}^{d}|c_j|^2 = 1$ and $\Sigma_{k=1}^{d}|e_k|^2 = 1$. After the PBS, the state of modes c and d reads

$$|\Psi_d\rangle = \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{c_j e_k}{\sqrt{(d-j)!(j-1)!(d-k)!(k-1)!}} (\hat{a}_{c,H}^\dagger)^{d-k} \times (\hat{a}_{c,V}^\dagger)^{j-1} (\hat{a}_{d,H}^\dagger)^{d-j} (\hat{a}_{d,V}^\dagger)^{k-1} |\text{vac}\rangle, \tag{12}$$

where, for example, $\hat{a}_{c,H}^\dagger$ denotes the photon creation operator of horizontal polarization in the spatial mode c. We consider only those events where the quantum nondemolition (QND) measurement device detects exactly $d-1$ photons in the output mode d. After this projection measurement, the output state of modes c and e reads

$$|\Psi_{\text{out}}\rangle = \sum_{j=1}^{d} f_j |j\rangle_c |j\rangle_e, \tag{13}$$

where $f_j = c_j d_j$. By using the setup of Fig. 2, one can prepare two entangled qudits of the form (13) with arbitrary f_j.

Supersinglets are states of total spin zero of N particles [each particle has spin of $(d-1)/2$] and are N-lateral rotationally invariant [12,21,22]. They can be used to solve some problems that have no classical solution. Supersingets of the form $|S_2^{(d)}\rangle$ (two-particle, d-level) can easily be prepared by using the setup of Fig. 2. First, the value of f_j in Eq. (13) is set to be $f_j = (-1)^j$. Then, the mode e passes through a half-wave plate which rotates its polarization by 90°, namely, $H \to V$ and $V \to H$. After these transformations, the final state is

$$|\Psi_{\text{out}}'\rangle = \sum_{j=1}^{d} (-1)^j |j\rangle_c |d+1-j\rangle_e. \tag{14}$$

State (14) is exactly the supersinglets state of two qudits [21].

Quantum entangled states of N qudits can also be generated by using N qudits (6) and the series of transformations of Fig. 2 (see Fig. 3). The input state can be given by

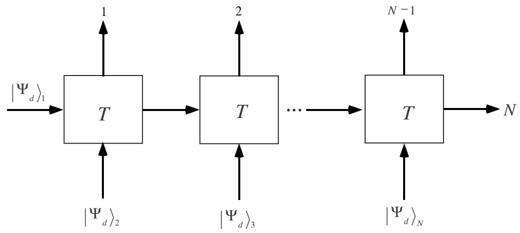

FIG. 3. Schematic setup for the generation of N entangled qudits. T, basic building block which represents the scheme shown in Fig. 2.

$$|\Psi_{\text{in}}\rangle = \sum_{j_1=1}^{d}\sum_{j_2=1}^{d}\cdots\sum_{j_N=1}^{d} c_{j_1}^{1}c_{j_2}^{2}\cdots c_{j_N}^{N}|j_1\rangle|j_2\rangle\cdots|j_N\rangle. \quad (15)$$

After a series of projection measurements, the output state reads

$$|\Psi_{\text{out}}\rangle = \sum_{j=1}^{d} c_j^1 c_j^2\cdots c_j^N|j\rangle_1|j\rangle_2\cdots|j\rangle_N = \sum_{j=1}^{d} C_j|j\rangle_1|j\rangle_2\cdots|j\rangle_N, \quad (16)$$

where C_j are arbitrary complex numbers. We have shown that, starting with single-photon Fock states, quantum entangled states of two qudits and even multiqudits can be conveniently produced by using linear optics and photon number projection measurements. It should be noted that the protocols proposed above can only be used to generate frequency-degenerate entangled states. However, it is highly desirable to build a source of two-color (two frequencies) or even multicolor entangled qudits in the quantum-information field. Figure 4 is a schematic setup to prepare a two-color quantum entangled state of two qudits.

First, two qudits of the form (6) with different frequencies ω_1 and ω_2 are injected into two input ports of a broadband polarizing beam splitter (here, broadband means that the polarizing characteristics are identical for both light fields of frequencies ω_1 and ω_2). The input state can be given by

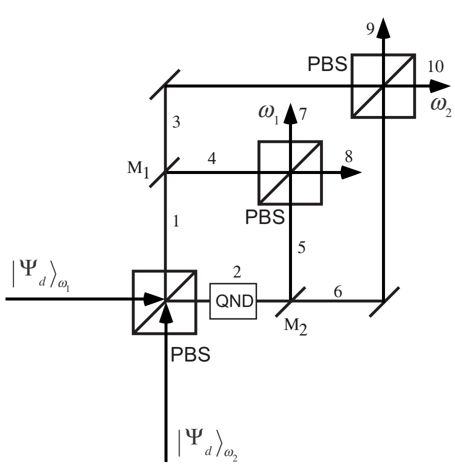

FIG. 4. Schematic setup for the generation of two two-color entangled qudits. M_1 and M_2 are dichroic mirrors.

$$|\Psi_{\text{in}}\rangle = |\Psi_d\rangle_{\omega_1}\otimes|\Psi_d\rangle_{\omega_2} = \sum_{j=1}^{d}\sum_{k=1}^{d} c_j e_k|j_{\omega_1}\rangle|k_{\omega_2}\rangle. \quad (17)$$

After the transformation of the PBS and considering only those cases where the QND device detects exactly $d-1$ photons (here the QND device is also assumed to be a broadband one, i.e., frequency insensitive), the postselected state of modes 1 and 2 has the form

$$|\Psi_{12}\rangle = \sum_{j=1}^{d} c_j e_j|(d-j)H_{\omega_2},(j-1)V_{\omega_1}\rangle_1 \otimes |(d-j)H_{\omega_1},(j-1)V_{\omega_2}\rangle_2. \quad (18)$$

Here the qudits have been expressed by the initial polarization states; $|(d-j)H_{\omega_1},(j-1)V_{\omega_2}\rangle$ means a single-mode quantum state with $d-j$ horizontal polarization photons of frequency ω_1 and $j-1$ vertical polarization photons of frequency ω_2, and so on.

Then, mode 1 is directed to M_1 and split up into modes 3 and 4, and mode 2 is directed to M_2 and split up into modes 5 and 6. Here, M_1 and M_2 are dichroic beam splitters that reflect light of frequency ω_1 and transmit light of frequency ω_2. The resulting state of modes 3, 4, 5, and 6 reads

$$|\Psi_{3456}\rangle = \sum_{j=1}^{d} e^{i(d-1)\tau_2}c_j e_j|(d-j)H_{\omega_2}\rangle_3|(j-1)V_{\omega_1}\rangle_4 \otimes |(d-j)H_{\omega_1}\rangle_5|(j-1)V_{\omega_2}\rangle_6, \quad (19)$$

where τ_2 is the time delay between arms 2 and 1. Now modes 3 and 6 have the same frequency ω_2 and, by combining them at a PBS, all the photons contained in the modes 3 and 6 are transferred to a single mode (mode 10) faithfully. Similarly, modes 4 and 5 have the same frequency ω_1 and, by combining them at a PBS, all the photons contained in the modes 4 and 5 are also transferred to a single mode (mode 7) faithfully. After these transformation and projection measurements, the state of modes 7 and 10 reads

$$|\Psi_{7,10}\rangle = \sum_{j=1}^{d} Pc_j e_j|(d-j)H_{\omega_1},(j-1)V_{\omega_1}\rangle_7 \otimes |(d-j)H_{\omega_2},(j-1)V_{\omega_2}\rangle_{10}, \quad (20)$$

where $P=e^{i(d-j)(\tau_2+\tau_5)}e^{i(j-1)(\tau_2+\tau_6)}$, τ_5 is the time delay between arms 5 and 4, and τ_6 is the time delay between arms 6 and 3. It is evident from Eq. (20) that the existence of P will inevitably introduce a phase modulation for the state (20). To eliminate such effects, we should have $\tau_2+\tau_5=2m\pi$ and $\tau_2+\tau_6=2n\pi$; here m and n are integers. Under such conditions, we have

$$|\Psi_{7,10}\rangle = \sum_{j=1}^{d} f_j|j_{\omega_1}\rangle_7|j_{\omega_2}\rangle_{10}, \quad (21)$$

where $f_j=c_j e_j$. It is evident that the state (21) is a two-color quantum entangled state of two qudits with arbitrary coefficients of f_j. In principle, one can realize N-color quantum entangled states of N qudits by repeating the transformation of Fig. 4 $N-1$ times. The setup is similar to that of Fig. 3; the basic building block T now represents the scheme shown in Fig. 4 and qudits of the form (6) with different frequencies are used as the input states. If some of the building blocks are replaced by the scheme shown in Fig. 3, a general h-color quantum entangled states of N qudits can be created ($1 \leqslant h \leqslant N$).

The main difficulty of the proposed schemes is the QND measurement of photon number at the single-photon level. In principle, such QND measurements can be fulfilled by Kerr-effect cross-phase-modulation between signal and probe beams and homodyne detection of the probe beam [23–25]; however, the experimental realization is still a challenge at present. An alternative method for the QND measurement shown in Fig. 2 is postselection measurement by photon detectors, just like those of many multiphoton entangled state experiments by SPDC. As the photons are recombined afterward, the QND measurement shown in Fig. 4 is necessary. To avoid the QND measurement, the generation of a two-color quantum entangled states of two qudits can also be achieved by using the scheme of Fig. 2 and inserting a polarization-independent frequency conversion device [26] at one output port of the PBS.

In conclusion, we have shown that arbitrary qudits can be prepared by carefully designing the polarization states of $d-1$ identical and independent photons and transferring them into a single spatial and temporal mode. Then, starting with N such qudits, N entangled qudits can be produced by linear optics and proper projection measurements. The frequencies of the N qudits can be identical or different according to the corresponding linear optics and projection measurement process.

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